A Derivation of Friis' Noise Factor Cascade Formula

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The definition of noise factor F_m is

$F_m = (S_{im} / N_{im}) / (S_{om} / N_{om})$,

where S denotes signal, N denotes noise power, i denotes input, o denotes output, m denotes the m–th amplifier, and where

$N_{im} = N_i$.

$N_i = kT_0B$ when the input impedance of the amplifier is equal to the thermal input noise source R_s,

where $k = 1.38 \times 10^{-23}$ J/°K is Boltzmann's constant, T_0 is 290°K , and B is the system noise power bandwidth in Hertz.

If the input impedance of the amplifier is R_i and not equal to R_s , then

$N_i = 4kT_0B\, R_s R_i/(R_s + R_i)^2$.

Note that $N_i(\text{watts}) = 3.98 \times 10^{-21}$ B watts, so that

$N_i(\text{dBm}) = 10 \log(1000 \times 3.98 \times 10^{-21} \times B) = -174 + 10 \log(B)$ dBm

when the input impedance of the amplifier is equal to the thermal input noise source R_s.

By algebraic rearrangement of the noise factor definition,

$F_m = N_{om} / [(S_{om}/S_{im}) N_i]$, so that
$F_m = N_{om} / (G_m N_i)$, where G_m is the gain of the m–th amplifier, so that
$F_m = (G_m N_i + N_m) / (G_m N_i)$, where N_m is the additional (excess) noise power added by the m–th amplifier.

By algebraic rearrangement of the last equation above,

$N_m = (F_m - 1)(G_m N_i)$.

In general, the total output noise power N_m(total) of the m-th amplifier is

$N_m(\text{total}) = (F_m - 1)(G_m N_i) + G_m N_i = F_mG_m N_i$, or

N_m(total) (dBm) = NF + 10 log(G) +10 log(B) – 174, where NF is the noise figure of the amplifier.

Next,

$(F_1 - 1)(G_1 N_i) = N_1$,

$(F_2 - 1)(G_2 N_i) = N_2$, and

$(F_c - 1)(G_c N_i) = N_c$,

where c denotes the cascade of amplifier 1 followed by amplifier 2.

The gain G_c of the cascade is

$G_c = G_1 G_2$,

and the cascaded noise N_c due to the two amplifiers alone is

$N_c = G_2 N_1 + N_2$.

It follows from the last formula, the next to the last formula, and the three formulas immediately above it that

$(F_c - 1)(G_1 G_2 N_i) = (F_1 - 1)(G_1 G_2 N_i) + (F_2 - 1)(G_2 N_i)$.

From the above it follows by dividing both sides of the equation by $G_1 G_2 N_i$ that

$F_c - 1 = F_1 - 1 + (F_2 - 1) / G_1$, or

$F_c = F_1 + (F_2 - 1) / G_1$.

If n amplifiers are cascaded together, then from the previous formula it can be shown that

$F_c = F_1 + (F_2 - 1) / G_1 + (F_3 - 1) / (G_1 G_2) + ... + (F_n - 1) / (G_1 G_2 ... G_{n-1})$.