Nyquist Noise Theory

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When I recently became interested in Nyquist's theory of Johnson noise, one of the first on line developments I came across was here . There it was stated that the power flow density of a resistor R is

$$P(f) = hf/(e^{hf/kT} - 1)\,df$$

where h is Planck's constant, f is frequency, k is Boltzmann's constant, and T is temperature. The formula above also follows immediately from the equivalent voltage formula in Nyquist's original article, “Thermal agitation of electrical charge in conductors,” **Physical Review**, 32, July 1928, 110 – 113. The remainder of this development is mine alone.

If the noise power output of a resistor is measured using a filter with arbitrarily steep skirts at frequencies f_1 and f_2 , $0 < f_1 < f_2$, and $B = f_2 - f_1$, then according to this theory the measured noise power should be

$$P = \int_{f_1}^{f_2} hf/(e^{hf/kT} - 1)\,df \ .$$

Using the “x substitution” $x = hf/kT$, the integral above becomes

$$P = kT(kT/h) \int_{f_1 h/kT}^{f_2 h/kT} x/(e^x - 1)\,dx .$$

Since

$$x < x + x^2/2! + x^3/3! + \ldots = e^x - 1 \quad \text{when} \quad 0 < x \ ,$$

it follows that

$$x/(e^x - 1) < 1 \quad \text{when} \quad 0 < x \ ,$$

from which it follows by integrating both sides of the inequality immediately above that

$$P \le kTB \ .$$

By comparing the two series

$$(e^x - 1)/x = 1 + x/2! + x^2/3! + \ldots \ \text{, and} \quad e^{0.0001} = 1 + 0.0001 + (0.0001)^2/2! + \ldots$$

term by term when $0 < x < 0.0001$, it follows that

$$(e^x - 1)/x < e^{0.0001} \ \text{, or} \quad e^{-0.0001} < x/(e^x - 1) \quad \text{when} \quad 0 < x \le 0.0001 \ ,$$

from which it follows by integrating both sides of the inequality immediately above that

$$e^{-0.0001} kTB \le P \ \text{, or} \quad 0.9999\, kTB \le P$$

for frequencies up to about 1 GHz. For lower frequencies the lower bound is better.

Consequently

$$P \approx kTB \ .$$

The power P is the power delivered to a matched load and is independent of the value of the noise (source) and load resistors R (if and only if both resistors have the same value R). From this it follows that the open source voltage V of a resistor R is

$$V \approx \sqrt{4\,kTRB} \ .$$